Characterizing Inefficiency of Infinite–Horizon Programs in Nonsmooth Technologies

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1. INTRODUCTION

An important problem in the theory of allocation of resources over an infinite time horizon is to find easily applicable criteria that can characterize the set of efficient (alternatively, inefficient) programs.

Restricting our attention to the standard aggregative model of economic growth, we find that there are two categories of results relating to this problem in the existing literature. One category of results relates to some partial characterizations of inefficiency under fairly general conditions on the technology (specifically, the production function f satisfies $f(0) = 0$; f is increasing; f is concave; f is continuous). On the necessity side, we have the well-known result of Malinvaud [5] that for an inefficient program the value of input is bounded away from zero. On the sufficiency side, we know that if the sequence of the value of consumption is summable, and the value of input is bounded away from zero, then the program is inefficient. These, and other related results, are discussed[1] in Section 3.

[1] Throughout the paper results (or minor variants of these) that are available in the existing literature are stated without proofs. In each case appropriate references where the interested reader can find complete proofs, are cited.

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It is quite clear that these necessity and sufficiency results are weak in content, since the class of interesting programs that can lie "in between" is rather large. However, for special technologies, which ensure that for every efficient program the sequence of the value of consumption is summable, these results provide a complete characterization of inefficiency.[2]

A second category of results relate to complete characterizations of inefficiency under fairly restrictive conditions on the technology (specifically, the curvature of the production function has to satisfy some uniformity requirements). These include the result of Cass [3] that an interior program is inefficient if and only if the sequence of the reciprocals of prices is summable. This result and its extensions by Benveniste and Gale [2] and Mitra [6] are surveyed in Section 4.

The restrictive assumptions under which these results are obtained mean that under more general circumstances they become inapplicable. More precisely, as two examples by Cass [3] demonstrate, the necessity results break down when the production function is "flat" for some range (without being flat for all ranges); and when the production function has a "kink," then the sufficiency result does not obtain.

It seems sensible, then, in view of the shortcomings of these two categories of theorems, to try to steer an intermediate course. That is, one might choose an important class of technologies and try to solve the problem stated in the first paragraph for these technologies only, but under fairly general assumptions in all other respects. This is precisely what we attempt in Section 5. We focus our attention there on "golden-rule technologies" (that is, those which admit of a golden-rule program) since this is certainly the most important and widely accepted set of technologies. It is, of course, clear that the results we expect to obtain now will be stronger than the first category of results and weaker than the second.

On the necessity side, we find that if a program is inefficient, then (a) there is a sequence of periods for which the input level of this program exceeds the golden-rule input level, and (b) the sum of the value of the difference of the input level of this program from the golden-rule input level, for the sequence of periods referred to in (a), is divergent (Theorem 5.1). The sufficiency result may be stated as follows. First, we define a variable z_t to represent the difference of the program's input level at time t from the golden-rule input level, if the former exceeds the latter; otherwise, it simply represents the

[2] It has been shown in Mitra [7] that under (A.1)–(A.3) and (A.4*), every efficient program has its sequence of the value of consumption summable if and only if (A.5) is satisfied. Since (A.5) excludes many interesting technologies (for example, the "golden-rule technologies" discussed in Section 5), the scope of the general necessity and sufficiency results, in providing complete characterizations of inefficiency, is clearly limited.

program's input level at time t. Then we define an "approximate price sequence," which can be made as close as we like to the usual (competitive) price sequence. If the value of z_t at such a price sequence is bounded away from zero, the program is inefficient.[3]

In providing these results in terms of the (appropriate value of the) difference of the program's input level from the golden-rule input level, we are following the lead of the famous Phelps–Koopmans theorem. In fact, a corollary of the sufficiency theorem is a simple proof of this well-known result. But there is other evidence that convinces us that these characterizations are useful. Specifically, we examine the two examples of Cass [3], referred to earlier, where (1) the results of Section 3 do not yield enough information to characterize the given programs, and (2) the results of Section 4 cannot be applied, since the assumptions under which they are proved are not satisfied. We find that our results characterize the programs in these examples very easily.

In Section 6 we look at a special case of a golden-rule technology, namely, one that is piecewise linear. This is of interest, since we are confronted with "flats" and a "kink" in the technology simultaneously, in the simplest way. The intuition gained from studying this case might provide characterizations in general "open" polyhedral models, which are scarce in the literature. Utilizing the simple structure of this technology, we are able to strengthen considerably the results of Section 5 (cf. Theorems 6.1, 6.2). The conditions of the two theorems are seen to be fairly close, but not equivalent, and this is precisely demonstrated by two examples of programs that lie "in between."

2. THE MODEL

Consider a one-good economy with a technology given by a function f from R^+ to itself. The production possibilities consist of inputs x and outputs $y = f(x)$ for $x \geq 0$.

The following assumptions on f will be used:

(A.1) $f(0) = 0$.

(A.2) f is strictly increasing for $x \geq 0$.

(A.3) f is concave for $x \geq 0$.

(A.4) f is continuous for $x \geq 0$.

[3] For a precise definition of two types of "approximate price sequences," see (3.8) and (5.4). For an accurate statement of the sufficiency theorem discussed here, see Theorem 5.2.

Under our assumptions, for every $x > 0$, there exists a left-hand derivative of f, denoted by $h(x)$. Also, for every $x \geq 0$, there exists a right-hand derivative of f, denoted by $g(x)$. By (A.2), $h(x) > 0$ for $x > 0$, and $g(x) > 0$ for $x \geq 0$. [$g(0)$ can, of course, be infinite.]

The initial input $\mathbf{x}$ is considered to be historically given, and positive. A *feasible production program* is a sequence $(x, y) = (x_t, y_{t+1})$ satisfying

$$x_0 = \mathbf{x}, \qquad x_t \leq y_t \quad \text{for} \quad t \geq 1, \qquad f(x_t) = y_{t+1} \quad \text{for} \quad t \geq 0 \tag{2.1}$$

The *consumption program* $c = (c_t)$ generated by (x, y) is defined by

$$c_t = y_t - x_t \,(\geq 0) \qquad \text{for} \quad t \geq 1. \tag{2.2}$$

(x, y, c) is called a *feasible program*, it being understood that (x, y) is a production program, and c is the corresponding consumption program.

A feasible program (x', y', c') *dominates* a feasible program (x, y, c) if $c'_t \geq c_t$ for all $t \geq 1$, and $c'_t > c_t$ for some t. A feasible program (x, y, c) is *inefficient* if there is a feasible program that dominates it. A feasible program is *efficient* if it is not inefficient.

We will associate with a feasible program (x, y, c) a *price sequence* $p = (p_t)$, given by[4]

$$p_0 = 1, \qquad p_{t+1} h(x_t) = p_t \qquad \text{for} \quad t \geq 0 \tag{2.3}$$

At these prices the feasible program (x, y, c) maximizes *intertemporal profits* at each date:

$$p_{t+1} f(x_t) - p_t x_t \geq p_{t+1} f(x) - p_t x, \qquad x \geq 0,\ t \geq 0 \tag{2.4}$$

The *value of input* sequence $v = (v_t)$ associated with a feasible program (x, y, c) is given by

$$v_t = p_t x_t \qquad \text{for} \quad t \geq 0 \tag{2.5}$$

A feasible program (x, y, c) is called *interior* if $\inf_{t \geq 0} x_t > 0$.

3. SOME GENERAL CHARACTERIZATION RESULTS

If we restrict ourselves to the minimal set of assumptions (A.1)–(A.4), what conditions can tell us whether a feasible program is inefficient or not?

[4] If $x_t = 0$ for some t, then we define p_{t+1} by the equation $p_{t+1} g(x_t) = p_t$, provided $g(0)$ is finite. If $g(0)$ is infinite, we follow the convention that $p_s = 0$ for $s > t$. It should be noted that given (A.1), feasible programs for which $x_t = 0$ for some finite t are not of much interest to us, since they terminate at a finite time period. This is why these details are not included in definition (2.3).

This section is devoted to answering this question. A series of interconnected results, which represent either necessary or sufficient conditions of inefficiency, are presented below. The usefulness of these general results is demonstrated by applications to special cases, where stronger assumptions than (A.1)–(A.4) are used.

We start with the following characterization result due to Cass [3].

LEMMA 3.1 (Cass) *Under (A.1) and (A.2), a feasible program* (x, y, c) *is inefficient iff there is a sequence* (e_t) *and* $1 \leq t < \infty$, *such that*

$$0 < e_t < x_t \qquad \textit{for} \quad t \geq t \tag{3.1}$$

$$e_{t+1} = f(x_t) - f(x_t - e_t) \qquad \textit{for} \quad t \geq t \tag{3.2}$$

Lemma 3.1 represents a complete characterization of inefficiency. However, it is clearly difficult to apply this result directly to test the inefficiency of a given feasible program, as, in a sense, it is a "redefinition" of the concept of inefficiency. Its merit lies in providing the basis from which useful characterizations may be obtained. For example, the well-known necessity theorem of inefficiency, stated below, is a direct consequence of it.

THEOREM 3.1 (Malinvaud) *Under (A.1)–(A.4), if a feasible program* (x, y, c) *is inefficient, then*

$$\inf_{t \geq 0} p_t x_t > 0 \tag{3.3}$$

This theorem yields the following useful corollary:

COROLLARY 3.1 *Under (A.1)–(A.4), if a feasible program* (x, y, c) *satisfies*

$$\infty > \sum_{t=1}^{\infty} p_t c_t \geq \sum_{t=1}^{\infty} p_t c_t' \tag{3.4}$$

for every feasible program (x', y', c') *then it violates (3.3) and is efficient.*

Turning next to sufficient conditions of inefficiency, the following theorem seems to be the most general available result.

THEOREM 3.2 (Mitra) *Under (A.1)–(A.4), if a feasible program* (x, y, c) *satisfies*

$$p_t x_t > 0 \qquad \textit{for} \quad t \geq 0 \tag{3.5}$$

and

$$\sum_{t=1}^{\infty} (p_t c_t / p_t y_t) < \infty \tag{3.6}$$

then it is inefficient.

This result was established in [6] under (A.1)–(A.3) and the following additional assumption:

(A.4*) f is differentiable for $x \geq 0$.

The reader can check that the proof goes through when (A.4*) is replaced by (A.4). A useful corollary of this Theorem is

COROLLARY 3.2 *Under (A.1)–(A.4), if an efficient program satisfies*

$$\sum_{t=1}^{\infty} p_t c_t < \infty \tag{3.7}$$

then it violates (3.3) and satisfies (3.4) for every feasible program (x', y', c').

These general results can be used to obtain complete characterizations of inefficiency in special cases where additional useful properties of f are known. For example, suppose (A.1)–(A.3) and (A.4*) are satisfied. In addition, suppose the following assumption holds:

(A.5) f satisfies one of the following three conditions:

(i) $\inf_{x \geq 0} f'(x) > 1$.
(ii) $\sup_{x \geq 0} f'(x) < 1$.
(iii) $\sup_{x \geq 0} f'(x) = 1 = f'(\underline{x})$ for some $\underline{x} > 0$.

It was shown in [7] that under (A.1)–(A.3), (A.4*), and (A.5), every efficient program (x, y, c) satisfies (3.7). Hence, using Theorem 3.1 and Corollary 3.2, we have the following complete characterization:

THEOREM 3.3 *Under (A.1)–(A.3), (A.4*), and (A.5), a feasible program (x, y, c) is inefficient if and only if it satisfies (3.3).*

Particular cases of this theorem have been obtained by McFadden [4] and Benveniste [1]. In [4] the production function f is assumed to be linear; i.e., $f(x) = dx$, where $d > 0$, so that (A.1)–(A.3), (A.4*), and (A.5) are clearly satisfied. In [1] f is assumed to satisfy (A.1)–(A.3), (A.4*), and (A.5)(i).

The general necessity result (Theorem 3.1) and the general sufficiency result (Theorem 3.2) are clearly wide apart in their contents. That is, under (A.1)–(A.4), there are many feasible programs that satisfy (3.3) and are efficient, or that are inefficient and violate (3.6). Furthermore, they are not exactly "comparable," in the sense that the two statements do not yield a clear idea of the type of programs that lie "in between." Thus it seems worthwhile to have a statement of a sufficiency theorem of inefficiency, which can be directly compared and contrasted with (3.3). For this purpose the notion of an "approximate price sequence" is useful.

Given any θ, such that $0 < \theta < 1$, an *approximate price sequence of type I*, $q(\theta) = (q_t(\theta))$, is defined by[5]

$$q_0(\theta) = 1, \qquad q_{t+1}(\theta)h(x_t - \theta x_t) = q_t(\theta) \qquad \text{for} \quad t \geq 0 \tag{3.8}$$

Notice that as $\theta \to 0$, $q_t(\theta) \to p_t$ for $t \geq 0$. We now have the following sufficiency result.

THEOREM 3.4 *Under (A.1)–(A.4), if a feasible program (x, y, c) satisfies*

$$\inf_{t \geq 0} q_t(\theta)x_t > 0 \tag{3.9}$$

for some $0 < \theta < 1$, then it is inefficient.

Proof Suppose (x, y, c) satisfies (3.9) for some $0 < \theta < 1$. Then, there is $\hat{e} > 0$, such that $q_t(\theta)x_t \geq \hat{e}$ for $t \geq 0$. Suppose e is a number satisfying $0 < q_t(\theta)e < \theta\hat{e} \leq \theta q_t(\theta)x_t$. Then e' is well defined by $e' = f(x_t) - f(x_t - e)$, and, furthermore, e' satisfies the condition $0 < q_{t+1}(\theta)e' < \theta\hat{e} \leq \theta q_{t+1}(\theta)x_{t+1}$. To check this, note first that since $0 < e < \theta x_t$, $e' > 0$ by (A.2). Also,

$$\begin{aligned} q_{t+1}(\theta)e' &= q_{t+1}(\theta)[f(x_t) - e)] \\ &\leq q_{t+1}(\theta)h(x_t - e)e \leq q_{t+1}(\theta)h(x_t - \theta x_t)e \\ &= q_t(\theta)e < \theta\hat{e} \leq \theta q_{t+1}(\theta)x_{t+1}. \end{aligned}$$

Thus, if we define $e_0 = \frac{1}{2}\theta\hat{e}$, then e_{t+1}, for $t \geq 0$, is well defined by $e_{t+1} = f(x_t) - f(x_t - e_t)$. Furthermore, for $t \geq 0$, $0 < e_t < x_t$. Hence, by Lemma 3.1 (x, y, c) is inefficient. ∎

For a trivial application of Theorem 3.4, consider the case where f is linear; i.e., $f(x) = dx$ where $d > 0$. Then, if a feasible program (x, y, c) satisfies (3.3), it satisfies (3.9) for *every* $0 < \theta < 1$. Hence, it is inefficient by Theorem 3.4. Thus by Theorem 3.1 we obtain a complete characterization of inefficiency in terms of condition (3.3).

A nontrivial application of the Theorem is that it enables us to obtain the result proved in Benveniste [1]. Suppose (A.1)–(A.3), (A.4*), and (A.5)(i) are satisfied. Then, following [1], it can be shown that given any feasible program (x, y, c), we have $\sup_{t \geq 0} p_t x_t < \infty$. Suppose, now, that a feasible program satisfies (3.3); then, using the arguments in [1] again, it can be shown that there is $N < \infty$ and $n > 1$, such that $x_{t+N} \geq nx_t$ for $t \geq 0$. Hence, choosing $(1 - \theta) = (1/n)$, we have $x_{t+N}(1 - \theta) \geq x_t$ for $t \geq 0$, or $f'(x_{t+N} - \theta x_{t+N}) \leq f'(x_t)$ for $t \geq 0$. This means that for $T \geq N$, $q_T(\theta) \geq q_N(\theta)p_T$. Since (x, y, c) satisfies (3.3), it must satisfy (3.9) as well. Hence, by Theorem 3.4, it is

[5] The qualifications made in footnote 4 also apply to the price sequence $(q_t(\theta))$. The definition of such a price sequence is motivated by the technique of proof employed to obtain a complete characterization result in [1].

inefficient. Thus, by Theorem 3.1 we obtain a complete characterization of inefficiency in terms of condition (3.3).

4. COMPLETE CHARACTERIZATIONS IN SMOOTH TECHNOLOGIES

If we assume (A.1)–(A.3) and (A.4*), and, furthermore, impose additional conditions on the production f to ensure that its "curvature" behaves uniformly (these are called "smoothness conditions"), then we can obtain complete characterizations of inefficiency. Furthermore, such characterizations have the advantage that they are in terms of "observable magnitudes," i.e., those that can be calculated along a feasible program without knowing the function f itself.

Notice that it is really not important for the production function actually to have "curvature," but that the curvature should satisfy certain uniformity requirements. Thus, if f has positive "curvature" in some range, it should in all ranges. Similarly, if it has zero curvature in some range (that is, it is flat), then it should be flat everywhere (that is, be a linear production function).

The most important of the complete characterizations is the one presented by Cass [3] (1) because his result relates to an important class of production functions, and (2) because his technique of proof can be used to obtain general results, as the extensions by Benveniste and Gale [2] and Mitra [6] amply demonstrate.

Cass assumes that f satisfies, in addition to (A.1) and (A.2),

(C.1) f is twice continuously differentiable for $x \geq 0$.

(C.2) f is strictly concave, with $f'' < 0$ for $x \geq 0$.

(C.3) f satisfies the endpoint conditions: $0 \leq f'(\infty) < 1 < f'(\underline{x}) < \infty$ for some $\underline{x} > 0$.

His result can be stated as follows:

THEOREM 4.1 (Cass) *Under (A.1), (A.2), and (C.1)–(C.3), an interior program (x, y, c) is inefficient if and only if*

$$\sum_{t=0}^{\infty} (1/p_t) < \infty \tag{4.1}$$

In extending this result to a wider class of production functions, as well as a wider class of feasible programs, Benveniste and Gale [2] assume:

(B.1) f is twice differentiable for $x \geq 0$.

(B.2) There are positive numbers, E, E', Q, Q', such that

$$E \leq [f'(x)x/f(x)] \leq E', \qquad Q \leq [-f''(x)x^2/f(x)] \leq Q' \qquad \text{for} \quad x \geq 0$$

THEOREM 4.2 (Benveniste–Gale) *Under (B.1) and (B.2), a feasible program (x, y, c) is inefficient if and only if*

$$\sum_{t=0}^{\infty} (1/p_t x_t) < \infty \tag{4.2}$$

It should be noted that while there is considerable overlap between the functions treated by Cass and by Benveniste and Gale, there are clearly cases in which the result of Cass applies, but not that of Benveniste and Gale, and vice versa. Also, the characterizations obtained in Theorems 4.1 and 4.2 seem to be qualitatively different from the result obtained in Theorem 3.3. It therefore seems desirable to obtain a theorem that unifies these results, and this is accomplished in [6].

In order to state this result, we assume (A.1)–(A.3) and (A.4*), and define the share of primary factor in output as

$$W(x) = 1 - [f'(x)x/f(x)] \quad \text{for} \quad x > 0; \qquad W(x) = 0 \quad \text{for} \quad x = 0 \tag{4.3}$$

and consider the following smoothness condition on the feasible program (x, y, c):

CONDITION S For some $0 < m \leq M < \infty$ and $0 < \theta < 1$,

$$\begin{aligned} meW(x_t)/x_t &\leq \{[f(x_t) - f(x_t - e)]/ef'(x_t)\} - 1 \\ &\leq MeW(x_t)/x_t \qquad \text{for} \quad 0 < e < \theta x_t, \quad t \geq 0 \end{aligned}$$

THEOREM 4.3 (Mitra) *Under (A.1)–(A.3) and (A.4*), a feasible program (x, y, c) satisfying Condition* S *is inefficient if and only if it satisfies (3.3) and*

$$\sum_{s=0}^{\infty} [W(x_s)/p_s x_s] < \infty \tag{4.4}$$

It should be noted that the results of Cass, Benveniste–Gale, McFadden, and Benveniste (which have been discussed above) can be obtained as corollaries of Theorem 4.3. Furthermore, Theorem 4.3 also provides a complete characterization for certain production functions that satisfy (A.1)–(A.3), (A.4*), and the assumption

(A.6) $f'(x) > 1$ for $x \geq 0$, and $\inf_{x \geq 0} f'(x) = 1$.

The criteria proposed in the earlier theorems cannot provide a complete characterization of inefficiency for such production functions. For detailed analysis of these results, the reader is referred to [6].

5. PARTIAL CHARACTERIZATIONS IN GOLDEN-RULE TECHNOLOGIES

Suppose the production function f does not satisfy the differentiability assumption (A.4*) and the uniformity requirements on its curvature, discussed in Section 4; then the complete characterization results break down, as the examples in Cass [3, pp. 221–222], and Mitra [6, Section 5] demonstrate. So without these smoothness conditions, can we do any better than the rather weak partial characterizations of Section 3? As a matter of fact, we can, if we restrict ourselves to "golden-rule technologies"—and these are certainly the most important class of technologies one would like to discuss, in any case.

Consider, then, that f satisfies (A.1)–(A.4), and the following additional assumption:

(A.7) There is $\bar{x} < \infty$, such that $f(\bar{x}) = \bar{x}$; for $0 < x < \bar{x}$, $x < f(x) < \bar{x}$; for $x > \bar{x}$, $x > f(x) > \bar{x}$.

When (A.7) holds, we call $\bar{x}$ the maximum sustainable input level. Let $C = [c \in R : c = f(x) - x$, where $0 \leq x \leq \bar{x}]$. Then $C \subset R^+$, C is nonempty, closed, and bounded. Hence there is $c^* \in C$ such that $c^* \geq c$ for all $c \in C$. By (A.7), $c^* > 0$. Consider, next, the set $X = [x \in R^+ : f(x) - x = c^*]$. X is nonempty by the way c^* was defined. Also, X is closed and bounded. Hence there is $x^* \in X$, such that $x^* \leq x$ for all $x \in X$. We call x^* the golden-rule input level and c^* the golden-rule consumption level. Since under (A.1)–(A.4) and (A.7) we can ensure the existence of a golden-rule input level, we refer to production functions satisfying (A.1)–(A.4) and (A.7) as "golden-rule technologies." Notice that the way x^*, c^*, were defined ensure that $g(x^*) \leq 1$, and given any $e > 0$, $g(x^* - e) > 1$. Also, (A.7) ensures that $h(\bar{x}) < 1$.

Given any feasible program (x, y, c), the assumptions (A.1)–(A.4) and (A.7) ensure that x_t, y_{t+1}, $c_{t+1} \leq \max(\mathbf{x}, \bar{x})$ for $t \geq 0$. We associate with each feasible program (x, y, c) a sequence $z = (z_t)$, defined for $t \geq 0$ by

$$z_t = (x_t - x^*) \quad \text{if} \quad x_t > x^*, \qquad z_t = x_t \quad \text{if} \quad x_t \leq x^* \tag{5.1}$$

We can now proceed to strengthen the partial characterizations obtained in Section 3. The main point to be noted about these stronger characterizations is that they are in terms of the value of the difference of the input level of the given feasible program from the golden-rule input level. In this respect, we are of course following the lead of the famous Phelps–Koopmans theorem. In fact, we shall see that a corollary of our sufficiency condition of inefficiency is this well-known theorem. On the necessity side, we make use of the simple observation that the *existence* of a golden-rule input level ensures that there is *some* curvature of the production function near the golden-rule input level, and apply methods similar to those used when such curvature is assumed to be present to begin with—for example, in the work of Cass [3].

We start with the necessity theorem of inefficiency.

THEOREM 5.1 *Under (A.1)–(A.4) and (A.7), if a feasible program (x, y, c) is inefficient, then*

(i) *condition (3.3) is satisfied;*
(ii) *the periods t_j for which $x_{t_j} > x^*$ are infinite in number;*
(iii)
$$\sum_{j=0}^{J} p_{t_j}(x_{t_j} - x^*) \to \infty \qquad as \quad J \to \infty. \tag{5.2}$$

Proof If (x, y, c) is inefficient, then (i) follows from Theorem 3.1. To prove (ii) and (iii), we note that by Lemma 3.1, there is a sequence (e_t) and $1 \le \tau < \infty$, such that (3.1) and (3.2) are satisfied.

Suppose, contrary to (ii), there is $\tau \le T < \infty$, such that $x_t \le x^*$ for $t \ge T$. Then using (3.1), we have for $t \ge T$, $e_{t+1} = [f(x_t) - f(x_t - e_t)] \ge h(x_t)e_t \ge e_t$. Hence $e_t \ge e_T > 0$ for $t \ge T$. Now clearly, $\{[f(x^*) - f(x^* - e_T)]/e_T\} = k > 1$. Otherwise, if $[f(x^*) - f(x^* - e_T)] \le e_T$, then $[f(x^*) - x^*] \le [f(x^* - e_T) - (x^* - e_T)]$, which contradicts the definition of the golden-rule input level. Hence for $t \ge T$,

$$\begin{aligned} e_{t+1} &= \{[f(x_t) - f(x_t - e_t)]/e_t\}e_t \\ &\ge \{[f(x^*) - f(x^* - e_t)]/e_t\}e_t \\ &\ge \{[f(x^*) - f(x^* - e_T)]/e_T\}e_t \\ &= ke_t, \end{aligned}$$

since $e_t \ge e_T$ and $x_t \le x^*$ for $t \ge T$. Thus $e_t \to \infty$ as $t \to \infty$, which contradicts (3.2), since $0 < e_t < x_t \le \max(\mathbf{x}, \bar{x})$. Hence (ii) must hold.

Suppose, contrary to (iii), that (5.2) is violated. By (3.1), we have $e_{t+1} = f(x_t) - f(x_t - e_t) \ge h(x_t)e_t$, so that $p_{t+1}e_{t+1} \ge p_t e_t$ for $t \ge \tau$. This means that for $t \ge \tau$, $p_t e_t \ge p_\tau e_\tau = b > 0$. Since (5.2) is violated, there is $T \ge \tau$, such that for $t_j \ge T$, we have $p_{t_j}(x_{t_j} - x^*) \le \frac{1}{2}b$. Hence for $t_j \ge T$,

$$\begin{aligned} e_{t_j+1} &= f(x_{t_j}) - f(x_{t_j} - e_{t_j}) \\ &\ge f(x^*) - f(x_{t_j} - e_{t_j}) \\ &= f(x^*) - f[x^* + (x_{t_j} - x^*) - e_{t_j}] \\ &\ge h(x^*)[e_{t_j} - (x_{t_j} - x^*)] \\ &\ge e_{t_j}[1 - \{p_{t_j}(x_{t_j} - x^*)/p_{t_j}e_{t_j}\}] \\ &\ge e_{t_j}[1 - \{p_{t_j}(x_{t_j} - x^*)/b\}]. \end{aligned}$$

For $t \geq T$ and $t \neq t_j$, we have $e_{t+1} = f(x_t) - f(x_t - e_t) \geq h(x_t)e_t \geq e_t$, since $x_t \leq x^*$. Hence for $t \geq 0$,

$$e_{t+T+1} \geq e_T \prod_{T \leq t_j \leq t+T} [1 - \{p_{t_j}(x_{t_j} - x^*)/b\}] \tag{5.3}$$

Since (5.2) is violated, (5.3) implies that there is $\hat{b} > 0$, such that $e_t \geq \hat{b}$ for $t \geq \tau$. Notice also that since, for $t \geq \tau$, $p_t e_t \geq b$, so $p_t x_t \geq b$. Hence, using the fact that $x_t \leq \max(\mathbf{x}, \bar{x})$, we have $\inf_{t \geq 0} p_t > 0$. This, in turn, implies that since $p_{t_j}(x_{t_j} - x^*) \to 0$ as $j \to \infty$, so $(x_{t_j} - x^*) \to 0$ as $j \to \infty$.

We observe that $\{[f(x^*) - f(x^* - \frac{1}{2}\hat{b})]/(\frac{1}{2}\hat{b})\} = \hat{k} > 1$. Otherwise, if $f(x^*) - f(x^* - \frac{1}{2}\hat{b}) \leq \frac{1}{2}\hat{b}$, then $f(x^*) - x^* \leq f(x^* - \frac{1}{2}\hat{b}) - (x^* - \frac{1}{2}\hat{b})$, which violates the definition of the golden-rule input. Choose θ such that $\frac{1}{2} < \theta < 1$ and $k' = \hat{k}\theta > 1$. Finally, choose $N \geq T$, such that for $t_j \geq N$, $(x_{t_j} - x^*) \leq \hat{b}(1 - \theta)$. Then, for $t_j \geq N$, we have

$$\begin{aligned}
e_{t_j+1} &= f(x_{t_j}) - f(x_{t_j} - e_{t_j}) \\
&\geq f(x^*) - f[x^* + (x_{t_j} - x^*) - e_{t_j}] \\
&\geq f(x^*) - f[x^* + e_{t_j}(1 - \theta) - e_{t_j}] \\
&= f(x^*) - f(x^* - \theta e_{t_j}) \\
&= \{[f(x^*) - f(x^* - \theta e_{t_j})]/\theta e_{t_j}\}\theta e_{t_j} \\
&\geq \{[f(x^*) - f(x^* - \theta\hat{b})]/\theta\hat{b}\}\theta e_{t_j} \\
&\geq \{[f(x^*) - f(x^* - \tfrac{1}{2}\hat{b})]/(\tfrac{1}{2}\hat{b})\}\theta e_{t_j} \\
&\geq \hat{k}\theta e_{t_j} = k' e_{t_j}.
\end{aligned}$$

For $t \geq N$ and $t \neq t_j$, we have already noted that $e_{t+1} \geq e_t$. Hence, using the fact that $k' > 1$ and the result of (ii), we have $e_t \to \infty$ as $t \to \infty$, which contradicts (3.2), since $0 < e_t < x_t \leq \max(\mathbf{x}, \bar{x})$. Hence (iii) must hold. ■

It was indicated earlier that a reason for obtaining the necessity result of Theorem 5.1 was that the characterization of Theorem 3.1 was too weak, while the characterizations of Section 4 were obtained under too strong a set of assumptions. The example of Cass [3, p. 221] illustrates this point very well, for there Theorem 3.1 does not yield enough useful information to judge the given feasible program to be efficient, while Theorems 4.1–4.3 give the wrong answer if they are used, since the assumptions under which they are proved no longer hold. Thus, a test of the usefulness of Theorem 5.1 is surely its capability of characterizing the feasible program of this example.

Following Cass, suppose f has a kink at x^*, such that $h(x^*) > 1 > g(x^*)$, and for $x \neq x^*$, f is differentiable. Also, f satisfies (A.1)–(A.4) and (A.7). Consider the feasible program (x, y, c) from $\mathbf{x} > x^*$, defined by $(x_{t+1} - x^*) =$

$\theta f'(x_t)(x_t - x^*)$ for $t \geq 0$, with $0 < \theta < 1$. Then, $x_t > x^*$, and

$$p_{t+1}(x_{t+1} - x^*) = \theta p_t(x_t - x^*)$$

for $t \geq 0$. Hence, $p_t(x_t - x^*) = \theta^t(\mathbf{x} - x^*)$ for $t \geq 0$, so that (5.2) is violated. Thus, by Theorem 5.1, (x, y, c) is efficient. Notice that since (3.3) is satisfied, so Theorem 3.1 is not able to characterize (x, y, c) as efficient. Also, (4.1), (4.2), and (4.4) are all satisfied, even though (x, y, c) is efficient.

We turn our attention, now, to the sufficiency theorem. For this purpose, as for Theorem 3.4, the notion of an "approximate price sequence" is useful. Given any θ, such that $0 < \theta < 1$, define an *approximate price sequence of type II*, $r(\theta) = (r_t(\theta))$, by[6]

$$r_0(\theta) = 1, \qquad r_{t+1}(\theta)h(x_t - \theta z_t) = r_t(\theta) \qquad \text{for} \quad t \geq 0 \tag{5.4}$$

THEOREM 5.2 *Under (A.1)–(A.4) and (A.7), if a feasible program (x, y, c) satisfies*

$$\inf_{t \geq 0} r_t(\theta)z_t > 0 \tag{5.5}$$

for some $0 < \theta < 1$, then it is inefficient.

Proof Suppose (x, y, c) satisfies (5.5) for some $0 < \theta < 1$. Then, there is $\hat{e} > 0$, such that $r_t(\theta)z_t \geq \hat{e}$ for $t \geq 0$. Suppose e is a number satisfying $0 < r_t(\theta)e < \theta\hat{e} \leq \theta r_t(\theta)z_t$. Then, e' is well defined by $e' = f(x_t) - f(x_t - e)$, and, furthermore, e' satisfies the condition $0 < r_{t+1}(\theta)e' < \theta\hat{e} \leq \theta r_{t+1}(\theta)z_{t+1}$. To check this, note first that since $0 < e < \theta z_t$, so by (A.2), $e' > 0$. Also,

$$\begin{aligned} r_{t+1}(\theta)e' &= r_{t+1}(\theta)[f(x_t) - f(x_t - e)] \\ &\leq r_{t+1}(\theta)h(x_t - e)e \leq r_{t+1}(\theta)h(x_t - \theta z_t)e \\ &= r_t(\theta)e < \theta\hat{e} \leq \theta r_{t+1}(\theta)z_{t+1}. \end{aligned}$$

Thus, if we define $e_0 = \frac{1}{2}\theta\hat{e}$, then e_{t+1}, for $t \geq 0$, is well defined by $e_{t+1} = f(x_t) - f(x_t - e_t)$. Also, $0 < e_t < z_t \leq x_t$ for $t \geq 0$. Hence by Lemma 3.1, (x, y, c) is inefficient. ■

It should be noted that when (3.9) is not satisfied, (5.5) may still be satisfied, and vice versa. Hence Theorem 5.2 is more useful in certain circumstances than Theorem 3.4. For example, suppose $f(x) = 2x^{1/2}$ for $x \geq 0$. Then, $x^* = 1$ and $\bar{x} = 4$. Consider the feasible program (x, y, c) from $[2(11/10)^4 - 1]$, given by $x_t = \{2[(t + 11)/(t + 10)]^4 - 1\}$ for $t \geq 0$. Choosing $\theta = \frac{1}{2}$, we note that $r_t(\theta) = [(t + 10)/10]^2$ for $t \geq 0$, and $z_t \geq [1/(t + 10)]$ for $t \geq 0$; so (5.5)

[6] The comments of footnote 4 apply to this price sequence as well.

is satisfied, and (x, y, c) is inefficient. However, since $x_t \to x^* = 1$ as $t \to \infty$, so given any θ, such that $0 < \theta < 1$, $q_t(\theta) \to 0$ as $t \to \infty$, violating (3.9).[7]

Another test of the usefulness of Theorem 5.2 is its capability of characterizing the program in the example of Cass [3, p. 222], where, once again, the results of Section 4 are inapplicable, since the assumptions under which they are established are violated. Consider, following Cass, that f satisfies (A.1)–(A.3), (A.4*), and (A.7), and has a flat for some range of $x \geq x^*$. More precisely, $f'(x) > 1$ for $0 \leq x < x^*$, $f'(x) = 1$ for $x^* \leq x \leq \bar{\bar{x}} < \bar{x}$, and $f'(x) < 1$ for $x > \bar{\bar{x}}$. The feasible program (x, y, c) given by $x_t = \mathbf{x}$, with $x^* < \mathbf{x} \leq \bar{\bar{x}}$, clearly satisfies (5.5) for any θ, such that $0 < \theta < 1$, and is, therefore, inefficient by Theorem 5.2. Notice, however, that the conditions (4.1), (4.2), and (4.4) are all violated by this program.

As a final test, we show that the Phelps–Koopmans theorem can be obtained as a corollary of Theorem 5.2. It should be noted that this result is proved in Phelps [8] and Cass [3] under stronger sets of assumptions than (A.1)–(A.4) and (A.7).

COROLLARY 5.1 (Phelps–Koopmans) *Under (A.1)–(A.4) and (A.7), if a feasible program (x, y, c) satisfies*

$$\liminf_{t \to \infty} x_t > x^* \tag{5.6}$$

then it is inefficient.

Proof If (x, y, c) satisfies (5.6), then there is $e > 0$, and $0 \leq t' < \infty$ such that $(x_t - x^*) \geq e$ for $t \geq t'$. Then, for any θ, such that $0 < \theta < 1$, we have $h(x_t - \theta z_t) = h[x_t(1 - \theta) + \theta x^*) \leq g(x^*) \leq 1$ for $t \geq t'$, since $x_t > x^*$ for $t \geq t'$. Hence $r_t(\theta) \geq r_{t'}(\theta)$ for $t \geq t'$, so that (5.5) is satisfied. Hence, by Theorem 5.2, (x, y, c) is inefficient. ■

6. A SPECIAL CASE OF FLATS AND KINKS

A case of interest among golden-rule technologies is a production function that is piecewise linear. In such a case the golden-rule input level occurs at a kink between two linear sections (alternatively, at a "switching point" between two techniques of production). Viewed as a special case of the production functions discussed in previous sections, it is of interest because here we have, simultaneously, the problem of the "flats" and that of the "kink." Viewed as a particular case of an "open" von Neumann model (where primary factors, exogenously supplied, limit production), it is of interest

[7] For an example of a case where Theorem 3.4 is more useful than Theorem 5.2, see Example 6.2 below.

since characterizations of inefficiency in open polyhdral models are relatively scarce in the literature.

We will consider the simplest case of piecewise linear technologies.[8] Specifically, we will assume that f satisfies.

(A.8) $f(x) = \min(ax, dx + n)$ for $x \geq 0$, where $a > 1$, $0 < d < 1$, $n > 0$.

Under (A.8), the golden-rule input level $x^* = [n/(a-d)]$, and the maximum sustainable input level $\bar{x} = [n/(1-d)]$.

If we define a capital-input-coefficients matrix $A = [n/(a-d), n/(1-d)]$, a labor-coefficients matrix $L = [1, 1]$, and a capital-output-coefficients matrix $B = [an/(a-d), n/(1-d)]$, and if the amount of labor exogenously supplied is stationary and normalized to unity, then the technological possibilities are given by the set

$$\mathcal{T} = [(x, y) \in R_+^2 : Az \leq x, Lz \leq 1, Bz \geq y, \quad \text{for some} \quad z \in R_+^2].$$

This is easily recognized as a simple open von Neumann model. The technological possibilities given by $\mathcal{T}$ coincide exactly with those specified by (A.8).

The necessity result of Theorem 5.1 [specifically, condition (iii)] can be strengthened under this simple structure, mainly because the curvature near the golden-rule input level can be more easily exploited.

THEOREM 6.1 *Under (A.8), if a feasible program (x, y, c) is inefficient, then*

(i) *condition (3.3) is satisfied;*
(ii) *the periods t_j, for which $x_{t_j} > x^*$, are infinite in number;*
(iii)
$$\limsup_{j \to \infty} p_{t_j}(x_{t_j} - x^*) > 0 \tag{6.1}$$

Proof Since (A.8) implies that (A.1)–(A.4) and (A.7) are satisfied, so (i) and (ii) follow from Theorem 5.1.

To prove (iii), suppose, on the contrary, that (6.1) is violated. Note that if (x, y, c) is inefficient, then there is a sequence (e_t) and $1 \leq \tau < \infty$, such that (3.1) and (3.2) are satisfied. Then $e_{t+1} = f(x_t) - f(x_t - e_t) \geq h(x_t)e_t$, so that $p_{t+1}e_{t+1} \geq p_t e_t$ for $t \geq \tau$. Hence $p_t e_t \geq p_\tau e_\tau = b > 0$ for $t \geq \tau$. Since (6.1) is

[8] It is possible to generalize from the simple case, involving two flat sections and a kink at the golden-rule input level, to any (finite) number of flat sections. The results become somewhat more complicated to state and are, therefore, omitted. It should be mentioned, in this connection, that since any smooth technology can be accurately approximated by piecewise linear technologies, it is of interest to know whether characterizations of inefficiency in the latter "approach" those in the former (in some appropriate sense) as individual flat sections become small, but the number of flats become infinitely large.

is violated, there is $T \geq \tau$, such that for $t_j \geq T$, we have $p_{t_j}(x_{t_j} - x^*) \leq \frac{1}{2}[(a-1)/(a-d)]p_{t_j}e_{t_j}$. Now, since $x_{t_j} > x^*$, so $f(x_{t_j}) = dx_{t_j} + n$; also, $[(x_{t_j} - x^*)/e_{t_j}] \leq \frac{1}{2}[(a-1)/(a-d)] < 1$; so $(x_{t_j} - e_{t_j}) < x^*$ and $f(x_{t_j} - e_{t_j}) = a(x_{t_j} - e_{t_j})$. Hence, for $t_j \geq T$,

$$\begin{aligned} e_{t_j+1} &= \{[f(x_{t_j}) - f(x_{t_j} - e_{t_j})]/e_{t_j}\}e_{t_j} \\ &= \{[f(x_{t_j}) - f(x^*) + f(x^*) - f(x_{t_j} - e_{t_j})]/e_{t_j}\}e_{t_j} \\ &= \{[d(x_{t_j} - x^*) + a(x^* - x_{t_j} + e_{t_j})]/e_{t_j}\}e_{t_j} \\ &= \{a - [(a-d)(x_{t_j} - x^*)/e_{t_j}]\}e_{t_j} \\ &\geq [a - \tfrac{1}{2}(a-1)]e_{t_j} = \tfrac{1}{2}(a+1)e_{t_j} \end{aligned}$$

since $(x_{t_j} - x^*) \leq \frac{1}{2}[(a-1)/(a-d)]e_{t_j}$. Also, for $t \geq T$, $t \neq t_j$, $e_{t+1} = f(x_t) - f(x_t - e_t) \geq h(x_t)e_t = ae_t$. Hence, for all $t \geq T$, $e_{t+1} \geq \frac{1}{2}(a+1)e_t$, which implies that $e_t \to \infty$ as $t \to \infty$. This contradicts (3.2), since $0 < e_t < x_t \leq \max(\mathbf{x}, \bar{x})$. ■

The usefulness of Theorem 5.2 is demonstrated, once again, as we can obtain from it a sufficiency theorem in terms of conditions that are "fairly close" to those proved in the necessity theorem above.

THEOREM 6.2 *Under (A.8), if for a feasible program* (x, y, c)

(i) *condition (3.3) is satisfied,*
(ii) *the periods t_j, for which $x_{t_j} > x^*$, are infinite in number, and*
(iii)
$$\liminf_{j \to \infty} p_{t_j}(x_{t_j} - x^*) > 0 \tag{6.2}$$

then (x, y, c) *is inefficient.*

Proof By (i)–(iii), there is $e > 0$, such that $p_t x_t \geq e$ for $t \geq 0$, and $p_{t_j}(x_{t_j} - x^*) \geq e$ for $j \geq 0$. Thus, for any θ, such that $0 < \theta < 1$, we have $h(x_t - \theta z_t) = h(x_t)$ for $t \geq 0$. Hence, for $t \geq 0$, $r_t(\theta)z_t = p_t z_t \geq e$, so that (5.5) is satisfied. Hence, by Theorem 5.2, (x, y, c) is inefficient. ■

The conditions in Theorems 6.1 and 6.2 are not equivalent.[9] To clarify the difference, we will exhibit programs that lie "in between." First, we give an example of an efficient program that satisfies the necessary conditions of inefficiency of Theorem 6.1. Then we give an example of an inefficient

[9] It is possible to obtain some stronger necessary conditions and weaker sufficiency conditions than those stated in Theorems 6.1 and 6.2, respectively. These are still not equivalent sets of conditions, and they add significantly to the complexity of the results; so we chose to omit them.

program, that does not satisfy the sufficient conditions of inefficiency of Theorem 6.2.

EXAMPLE 6.1 Let $a = 6, d = (1/3), n = (17/3)$. Then $x^* = 1, \bar{x} = (17/2)$. Consider the feasible program (x, y, c) from $\mathbf{x} = 2$, defined by $x_t = 2$ for t odd (called t_i) and by $x_t = 1 + (\frac{1}{4})^t$ for t even (called t_j). Hence $x_t > x^*$ for $t \geq 0$, so that $p_t = 3^t$ for $t \geq 0$. Thus conditions (i) and (ii) of Theorem 6.1 are satisfied. Also, $p_{t_i}(x_{t_i} - x^*) \to \infty$ as $t_i \to \infty$; so condition (iii) of Theorem 6.1 is also satisfied. However, (x, y, c) is efficient. To verify this, suppose it were not. Then, there is a sequence (e_t) and $1 \leq \tau < \infty$, such that (3.1) and (3.2) hold. Since $p_{t_j} z_{t_j} \to 0$ as $t_j \to \infty$, so by applying an argument identical to that in Theorem 6.1, there is T, such that for $t_j \geq T$, $z_{t_j} \leq \frac{1}{2}[(a-1)/(a-d)]e_{t_j}$ and $e_{t_j+1} \geq \frac{1}{2}(a+1)e_{t_j} = (\frac{7}{2})e_{t_j}$. For $t_i \geq 0$, we have $e_{t_i+1} \geq (\frac{1}{3})e_{t_i}$. Hence $e_t \to \infty$ as $t \to \infty$, which contradicts (3.2), since $0 < e_t < x_t \leq \max(\mathbf{x}, \bar{x})$.

EXAMPLE 6.2 Let $a = 3, d = (\frac{1}{3}), n = (\frac{8}{3})$. Then $x^* = 1, \bar{x} = 4$. Consider the feasible program (x, y, c) from $\mathbf{x} = 2$, defined by $x_t = 2$ for t odd (called t_i) and by $x_t = 1 + (\frac{1}{4})^t$ for t even (called t_j). Then $x_t > x^*$ for $t \geq 0$, so that $p_t = 3^t$ for $t \geq 0$. Thus conditions (i) and (ii) of Theorem 6.2 are satisfied. Also, $p_{t_j}(x_{t_j} - x^*) \to 0$ as $t_j \to \infty$; so condition (iii) of Theorem 6.2 is not satisfied. However the program is inefficient. To check this, let $\theta = (\frac{3}{4})$, and notice that $h(x_0 - \theta x_0) = (\frac{1}{3})$, $h(x_t - \theta x_t) = (\frac{1}{3})$ for t odd, and $h(x_t - \theta x_t) = 3$ for t even. Hence (3.9) is satisfied, so that by Theorem 3.4, (x, y, c) is inefficient. Incidentally, this example also illustrates the fact that Theorem 3.4 may be applicable in some cases where Theorem 5.2 is not applicable.

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